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On attenuation of plane sound waves in turbulent mean flow

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Abstract

Plane sound waves in a smooth pipe turbulent boundary layer are known to be more strongly damped when the acoustic boundary layer becomes thicker than the viscous sublayer. The attenuation constants that govern this phenomenon are accurately predicted by the mathematical model proposed by M.S. Howe [The damping of sound by wall turbulent shear layers. *Journal of the Acoustical Society of America* 98(3) (1995) 1725–1730. Also in: *Acoustics of Fluid–Structure Interactions*, Cambridge University Press, Cambridge, 1998]. This model assumes uniform mean core flow. The present paper proposes a variant of this model which is based on the assumption of parallel sheared mean core flow. Predictions of the two approaches are compared.

1. Introduction

The damping of plane sound waves in a smooth pipe carrying a fully developed turbulent mean flow has been the subject of several experimental and theoretical studies. An inclusive review of the previous work can be found in Peters et al. [1]. The experimental data reveal that, in the presence of a fully developed subsonic turbulent mean flow, the damping of plane sound waves becomes stronger when the acoustic boundary layers becomes thicker than the viscous sublayer. The analytical model proposed by Howe [2] appears to capture this effect quite accurately. Recent experiments by Allam and Åbom [3] re-confirm the accuracy of Howe's model. It is thus desirable to express this model in transfer matrix form for use in flow duct acoustics calculations. This seems to be readily achievable, since the governing dispersion relation is based on the continuity and axial momentum equations ($e^{-i\omega t}$ time dependence being assumed),

$$\frac{1}{c_0^2}\left(-i\omega + \frac{4\rho_0 Y}{D_p} + U\frac{\partial}{\partial x}\right)p + \rho_0\frac{\partial v}{\partial x} = 0, \tag{1}$$

$$\rho_0\left(-i\omega + U\frac{\partial}{\partial x}\right)v + \frac{\partial p}{\partial x} = 0, \tag{2}$$

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respectively, or the wave equation that results from these upon elimination of the axial component of the acoustic particle velocity, v, namely,

$$\frac{1}{c_0^2}\left(-\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)^2 p + \frac{4\rho_0 Y}{D_p}\left(-\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)p - \frac{\partial^2 p}{\partial x^2} = 0. \tag{3}$$

Here, Y denotes Howe's turbulent boundary layer admittance, which is defined as ratio of the normal component of the acoustic particle velocity at the border of the wall boundary layer to the acoustic pressure, and the remaining symbols have the usual meanings (ω denotes the radian frequency, t the time, i the unit imaginary number, x the duct axis, p the acoustic pressure, U the mean flow velocity, c_0 and ρ_0 the speed of sound and the mean density of the fluid, respectively, and D_p the hydraulic diameter of the duct). The reader is referred to Appendix A for the formulae used for the computation of Y and to Ref. [2] for its derivation. Howe [2] assumes a uniform core flow with a thin boundary layer so that the acoustic pressure and particle velocity terms in Eqs. (1)–(3) can be defined as average values over the duct cross-sectional area, rather than assuming them to be plane *ab initio*. The resulting dispersion equation can be solved only numerically, since Y depends on the wavenumbers, however, upon assuming that frequency is high enough, the wavenumbers are obtained in the approximate analytical form proposed by Howe [2]. This approximate solution can be used to construct a diagonal transfer matrix, but it will not be exactly consistent with Eqs. (1) and (2).

The dependence of Y on the wavenumbers of Eqs. (1) and (2) arises, because the acoustic field depicted by these equations is tacitly assumed to extend into the boundary layer. In the present analysis, this assumption is abandoned and the problem is modeled as that of plane sound wave propagation in a uniform pipe carrying a parallel sheared mean flow, the duct wall being assumed to obey Howe's admittance with the Kirchhoff wavenumbers [1]. This model yields a simple transfer matrix with closed expressions for the wavenumbers and the attenuation constants are in close agreement with Howe's model.

2. Analysis

The model adopted in the present analysis is a uniform duct carrying a parallel sheared mean flow, with possibly some slip flow of velocity w_0 at the wall and the wall obeying some impedance relationship. In this model, the mean flow represents the core flow in the actual duct; the slip flow corresponds to the mean flow at the edge of the boundary layer; and, it is assumed that, the acoustic phenomena in the boundary layer can be simulated by a wall impedance model.

The continuity and axial momentum equations which govern the propagation of plane sound waves in this uniform duct can be expressed as [4]

$$\frac{1}{c_0^2}\left(-\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)p + \rho_0\frac{\partial v}{\partial x} = \mu, \tag{4}$$

$$\rho_0\left(-\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)v + \left(1 + \frac{\beta U^2}{c_0^2}\right)\frac{\partial p}{\partial x} = (w_0 - U)\mu, \tag{5}$$

respectively. Here, μ denotes the fluctuating component of the rate of radial mass flow into the pipe per unit volume, β is a non-dimensional parameter that depends on the mean flow profile [4] and U is now understood as the cross-sectional average of the mean flow velocity. Since a subsonic low Mach number mean flow is of interest and $\beta < 0.1$ for turbulent mean flow profiles, $\beta U^2/c_0^2 \ll 1$ with less than about 1% error and, consequently, this term is neglected in the subsequent analysis. It should be noted that Eqs. (4) and (5) assume that the acoustic pressure and density are isentropically related. This assumption is not strictly true with sheared parallel mean flow, however, for subsonic low Mach numbers, it provides an accurate representation of the linearized energy equation for plane wave propagation [4].

The fluctuating mass inflow at the border of the boundary layer can be expressed as $\mu = -4\rho_0 u/D_p$, where u denotes the radial component of the particle velocity. Then, introducing the boundary layer admittance,

$\mu = -4\rho_0 Yp/D_p$. Substituting this in Eqs. (4) and (5) yields, respectively,

$$\frac{1}{c_0^2}\left(-\mathrm{i}\omega + \frac{4\rho_0 Y}{D_p} + U\frac{\partial}{\partial x}\right)p + \rho_0\frac{\partial v}{\partial x} = 0, \tag{6}$$

$$\rho_0\left(-\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)v + \frac{4Y\rho_0}{D_p}(w_0 - U)p + \frac{\partial p}{\partial x} = 0. \tag{7}$$

The slip velocity, w_0, is kept as a model parameter in the following analysis, although it will in general be small compared to U to justify the assumption of $w_0/U \approx 0$, or, $\phi \approx 1$ in Eq. (10), since the boundary layer is thin. On the same premise, D_p can still be taken as the pipe diameter. The wall impedance is given by Howe's boundary layer admittance, the impinging axial wave assumed to have the Kirchhoff wavenumbers for fundamental visco-thermal propagation with no mean flow [1]. With this assumption, which may be justified on the premise that visco-thermal effects dominate in the boundary layer, Eqs. (6) and (7) can be solved relatively easily. The effects of the mean flow gradient in the boundary layer and particle velocity incompatibility at the edge of the boundary layer are assumed to be negligible.

It is convenient to introduce the transformation

$$p = p^+ + p^-, \quad \rho_0 c_0 v = p^+ - p^-, \tag{8}$$

which, when substituted in Eqs. (6) and (7), yields

$$\frac{\partial}{\partial x}\begin{bmatrix} p^+ \\ p^- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \dfrac{\mathrm{i}2k_0 - (1-\phi M)a}{1+M} & -\dfrac{(1-\phi M)a}{1+M} \\ \dfrac{(1+\phi M)a}{1-M} & \dfrac{-\mathrm{i}2k_0 + (1+\phi M)a}{1-M} \end{bmatrix}\begin{bmatrix} p^+ \\ p^- \end{bmatrix}, \tag{9}$$

where

$$k_0 = \frac{\omega}{c_0}, \quad M = \frac{U}{c_0}, \quad \phi = 1 - \frac{w_0}{U}, \quad a = \frac{4\rho_0 c_0 Y}{D_p}. \tag{10}$$

The general solution of Eq. (9) can be expressed in transfer matrix form as

$$\begin{bmatrix} p^+(x) \\ p^-(x) \end{bmatrix} = \mathbf{\Phi}\begin{bmatrix} e^{\mathrm{i}K^+x} & 0 \\ 0 & e^{\mathrm{i}K^-x} \end{bmatrix}\mathbf{\Phi}^{-1}\begin{bmatrix} p^+(0) \\ p^-(0) \end{bmatrix}. \tag{11}$$

Here, K^+ and K^- denote the wavenumbers, which are the eigenvalues of the square matrix in Eq. (9) divided by the unit imaginary number, for acoustic wave motion in forward (with the mean flow) and backward directions, respectively, and $\mathbf{\Phi}$ is the modal matrix whose columns are the eigenvectors corresponding to the two wavenumbers.

The wavenumbers are:

$$K^{\pm} = \frac{\pm k_0}{1 \pm M}\,\frac{\mp M \mp \dfrac{\mathrm{i}aM(1+\phi)}{2k_0} + \sqrt{1 - \dfrac{a^2M^2(1+\phi)^2}{4k_0^2} + \dfrac{\mathrm{i}a(1+\phi M^2)}{k_0}}}{1 \mp M}. \tag{12}$$

The imaginary part of which determines the attenuation constants:

$$\alpha^{\pm} = \frac{1}{1-M^2}\left\{-\frac{1}{2}M(1+\phi)\,\mathrm{Re}(a) \pm k_0\,\mathrm{Im}\sqrt{1 - \frac{a^2M^2(1+\phi)^2}{4k_0^2} + \frac{\mathrm{i}a(1+\phi M^2)}{k_0}}\right\}. \tag{13}$$

The modal matrix is

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & -1 + \mathrm{i}2\dfrac{k_0 + K^-(1-M)}{(1+\phi M)a} \\ -1 + \mathrm{i}2\dfrac{k_0 - K^+(1+M)}{(1-\phi M)a} & 1 \end{bmatrix}. \quad (14)$$

A high frequency approximation, which is similar to that used in deriving the wavenumbers in Ref. [2], can be obtained from the first order expansion of the square root in Eq. (12):

$$K^{\pm} = \frac{\pm k_0}{1 \pm M} + \frac{\mathrm{i}a}{2}\frac{\pm 1 - \phi M}{1 \pm M}, \quad \alpha^{\pm} = \frac{1}{2}\frac{\pm 1 - \phi M}{1 \pm M}\mathrm{Re}(a), \quad \boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (15)$$

3. Numerical results and conclusion

Shown in Figs. 1 and 2 is the variation of the attenuation ratio $\alpha^{\mp}/\alpha_0$ with the normalized boundary layer thickness, δ_A^+, where α_0 denotes the Kirchhoff solution for the visco-thermal attenuation constant for the case

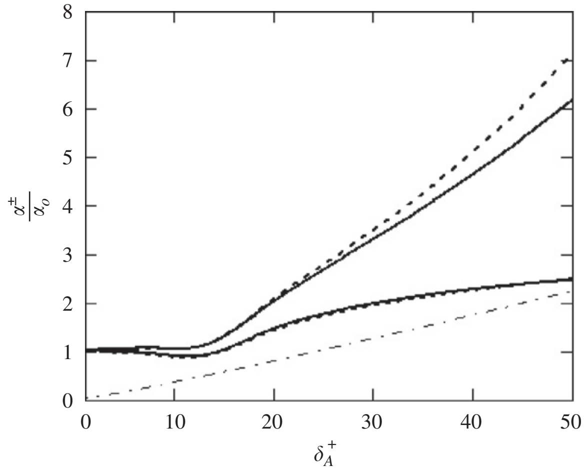

Fig. 1. The attenuation ratio for the $k_0R = 0.0323$ case tested in Ref. [3], $D_p = 35$ mm. Dashed curves are Howe's solution, solid curves are the present solution with $\phi = 1$. The upper curves correspond to the backward wave. The dash-dotted curve scales δ_A^+ with $10M$.

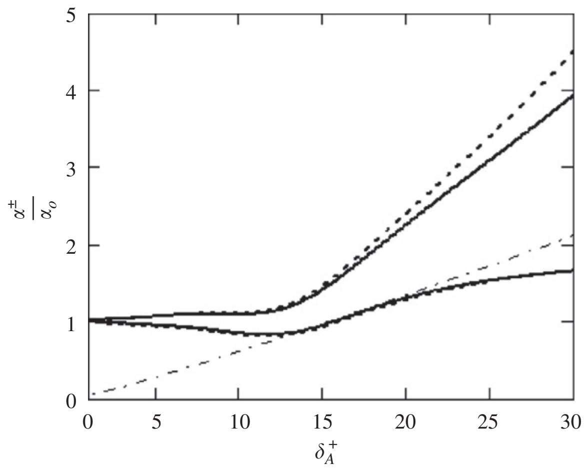

Fig. 2. The attenuation ratio for the $k_0R = 0.0808$ case tested in Ref. [3], $D_p = 35$ mm. Dashed curves are Howe's solution, solid curves are the present solution with $\phi = 1$. The upper curves correspond to the backward wave. The dash-dotted curve scales δ_A^+ with $10M$.

of zero mean flow, for two frequencies tested by Allam and Åbom [3]. The attenuation ratios are plotted as function of δ_A^+ for uniformity with their results. Expression for δ_A^+ and α_0 are given in Appendix A. δ_A^+ can be scaled with the average mean flow velocity Mach number M by using the dash-dot curves in Figs. 1 and 2.

The present solution for the forward wave almost coincides with Howe's solution; however, the values predicted for the attenuation constant of the backward wave are slightly smaller than Howe's solution, but look a little bit closer to the experimental results [3] for relatively large δ_A^+. The experimental points are not shown, because they could not be read with good accuracy from the logarithmic scales of the published figures.

It is noteworthy that the present results given Figs. 1 and 2 are for $\phi = 1$, that is, the no-slip flow case. As meaningful small slip flow is introduced, the separation of the forward and backward attenuation ratios tends to decrease slightly. In the limiting case of uniform mean flow, $\phi = 0$, however, the attenuation ratios become grossly inaccurate.

The sound field in the duct can be decomposed into the sum of forward and backward components if the modal matrix is diagonal. If the diagonal terms are normalized to unity for the cases considered in Figs. 1 and 2, the real and imaginary parts of the off-diagonal elements of $\mathbf{\Phi}$ are found to be close to zero within 2%. Thus, the plane acoustic wave field is not truly canonical, but the coupling can be considered small enough to be neglected for practical purposes in these cases.

Appendix A

Howe [2] boundary layer admittance can be expressed as

$$\frac{\rho_0 c_0}{R} Y = \frac{k_0}{s\sqrt{\mathrm{i}}}\left(K_0^2 F_v + \frac{\gamma - 1}{\sqrt{\sigma}} F_\chi\right). \tag{A.1}$$

Here, $R = D_p/2$, $s = \sqrt{\omega R^2/v}$ is the Stoke's number, γ denotes the ratio of the specific heat coefficients, $\sigma = \mu c_p/k$ the Prandtl number, μ the coefficient of shear viscosity, $v = \mu/\rho_0$ the kinematic viscosity, c_p the specific heat coefficient at constant pressure, κ the thermal conductivity, K_0 the propagation constant of the axial wave motion in the boundary layer, and functions F_v and F_χ are

$$F_v = F_0\left(\frac{\sqrt{\mathrm{i}}}{\kappa_K \delta}, \frac{\lambda\sqrt{\mathrm{i}}}{\delta}\right), \quad F_\chi = F_0\left(\frac{\sqrt{\mathrm{i}\sigma_t^2/\sigma}}{\kappa_K \delta}, \frac{\lambda\sqrt{\mathrm{i}\sigma}}{\delta}\right), \tag{A.2}$$

where $\delta = \delta_A^+/\sqrt{2} = u_*/\sqrt{\omega v}$, $\sigma_t \approx 0.7$ is turbulence Prandtl number for air [2], $\kappa_K \approx 0.41$ is the von Karman constant [2], δ_A^+ denotes the normalized boundary layer thickness [3], $A_{\mathrm{Re}} = \rho_0 R c_0/\mu$ is the acoustic Reynold's number, u_* the friction velocity, which is determined from Prandtl's logarithmic law

$$\frac{U}{u_*} = 2.0 + 2.44\left(\ln\, MA_{\mathrm{Re}} - \ln\frac{U}{u_*}\right), \tag{A.3}$$

and the function $F_0(x, y)$ is

$$F_0(x,y) = \mathrm{i}\frac{H_1^{(1)}(x)\cos y - H_0^{(1)}(x)\sin y}{H_1^{(1)}(x)\sin y + H_0^{(1)}(x)\cos y}, \tag{A.4}$$

where $H_n^{(1)}$ denotes Hankel function of the first kind of order n. In Eq. (A.2), the parameter λ defines the frequency dependence of the viscous sublayer thickness and is computed from the empirical equation [2]

$$\lambda = 6.5\left[1 + \frac{1.7\left(\frac{100}{\delta^2}\right)^3}{1 + \left(\frac{100}{\delta^2}\right)^3}\right]. \tag{A.5}$$

In the present analysis, K_0 is assumed to be given by the Kirchhoff solution for fundamental mode visco-thermal propagation [1], that is,

$$K_0 = 1 + \frac{1+\mathrm{i}}{\sqrt{2}} \frac{\gamma - 1 + \sqrt{\sigma}}{s\sqrt{\sigma}} \tag{A.6}$$

to first order in s. It is usually convenient to normalize the attenuation constants for the non-zero mean flow case with the attenuation constant for this case, namely, $\alpha_0 = \mathrm{Im}(k_0 K_0)$.

The normalized boundary layer thickness, δ_A^+ can be expressed as function of the mean velocity Mach number as

$$\delta_A^+ = \sqrt{\frac{2A_{\mathrm{Re}}}{k_0 R}} \frac{u_*}{U} M, \tag{A.7}$$

where u_*/U depends on M as depicted by Eq. (A.3).

References

[1] M.C.A.M. Peters, A. Hirshberg, A.J. Reinjnen, A.P.J. Wijnands, Damping and reflection coefficient measurements for an open pipe at low Mach and low Helmholtz numbers, *Journal of Fluid Mechanics* 256 (1993) 499–534.

[2] M.S. Howe, The damping of sound by wall turbulent shear layers, *Journal of the Acoustical Society of America* 98 (3) (1995) 1725–1730 Also in: *Acoustics of Fluid–Structure Interactions*, Cambridge University Press, Cambridge, 1998.

[3] S. Allam, M. Åbom, Investigation of damping and radiation using full plane wave decomposition in ducts, *Journal of Sound and Vibration* 292 (2006) 519–534.

[4] E. Dokumaci, Effect of sheared grazing mean flow on acoustic transmission in perforated pipe mufflers, *Journal of Sound and Vibration* 283 (2005) 645–663.